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The Schrödinger operator as a generalized Laplacian

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Abstract

The Schrödinger operators on the Newtonian spacetime are defined in a way which make them independent of the class of inertial observers. In this picture the Schrödinger operators act not on functions on the spacetime but on sections of a certain one-dimensional complex vector bundle—the *Schrödinger line bundle*. This line bundle has trivializations indexed by inertial observers and is associated with an $U(1)$-principal bundle with an analogous list of trivializations—the *Schrödinger principal bundle*. If an inertial frame is fixed, the Schrödinger bundle can be identified with the trivial bundle over spacetime, but as there is no canonical trivialization (inertial frame), these sections interpreted as 'wavefunctions' cannot be viewed as actual functions on the spacetime. In this approach, the change of an observer results not only in the change of actual coordinates in the spacetime but also in a change of the phase of wavefunctions. For the Schrödinger principal bundle, a natural differential calculus for 'wave forms' is developed that leads to a natural generalization of the concept of the Laplace–Beltrami operator associated with a pseudo-Riemannian metric. The free Schrödinger operator turns out to be the Laplace–Beltrami operator associated with a naturally distinguished invariant pseudo-Riemannian metric on the Schrödinger principal bundle. The presented framework does not involve any ad hoc or axiomatically introduced geometrical structures. It is based on the traditional understanding of the Schrödinger operator in a given reference frame—which is supported by producing right physics predictions—and it is proven to be strictly related to the frame-independent formulation of analytical Newtonian mechanics and Hamilton–Jacobi equations that makes a bridge between the classical and quantum theory.

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1. Introduction

In the papers [5–7, 24], we have presented an approach to differential geometry in which sections of a one-dimensional affine bundle over a manifold have been used instead of functions on the manifold. This approach, initiated by Tulczyjew *et al* [22, 23], has been successfully applied to a frame-independent description of different systems, in particular to a frame-independent formulation of Newtonian mechanics [10].

The latter problem is closely related to the problem of frame-independent formulation of wave mechanics in the Newtonian spacetime. It is known that a solution of the Schrödinger equation in one inertial frame will not, in general, satisfy the Schrödinger equation in a different frame. The same quantum state of a particle must be represented by a different wavefunction in reference to a different inertial frame. The corresponding gauge transformation of solutions of the Schrödinger equation was already known to Pauli [17]. A number of ways of solving this problem have been proposed in the literature. For instance, a general axiomatic theory of quantum bundles, quantum metrics, quantum connections, etc has been developed in [13] to deal with a covariant description of Schrödinger operators in curved spacetimes. Another general fibre bundle formulation of nonrelativistic quantum mechanics has been proposed in a series of papers [12].

An approach which is the closest to what we propose in this paper is a frame-independent formulation of wave mechanics by extending the Newtonian spacetime to five-dimensional Galilei space [3, 20, 21]. The corresponding geometry is associated with the Bargmann group—nontrivially extended Galilei group [1].

In the present paper we change this viewpoint a little bit, making the 'wavefunctions' living on a four-dimensional base again. For simplicity, we deal with the flat Newtonian spacetime and the very standard Schrödinger operators to show that a frame-independent formulation of wave mechanics (for every mass $m \neq 0$) is possible in terms of a principal $U(1)$ bundle P_m—the *Schrödinger principal bundle*. For a fixed inertial frame this bundle can be identified with the trivial bundle over the spacetime, but no canonical trivialization is given. With this bundle there is associated a complex line bundle L_m—the *Schrödinger line bundle*. Only the projective class of this bundle is uniquely defined, which is associated with the fact that wavefunctions are sometimes understood as defined up to a phase factor. In our picture, the Schrödinger operator acts not on functions on the spacetime but on sections of L_m. This bundle, constructed from the data provided by all possible inertial observers, has no canonical trivialization, so its sections cannot be viewed as functions on the spacetime. Indeed, they change under the change of an inertial frame in a way which is different from the way functions do. We would like to stress that this point often causes difficulties for some people who have problems with distinguishing trivializable bundles from trivial ones. This distinction should be taken seriously while reading this paper. One can simply explain this problem in plain English by pointing out that 'mortal' is not the same as 'dead'. One can interpret this fact in the way that passing to another observer leads not only to a certain change in positions and velocities but also to a change in the phase of wavefunctions.

Having constructed the Schrödinger principal bundle P_m as the proper geometrical tool for understanding the Schrödinger operators, we develop a differential calculus based on the *Atiyah Lie algebroid* $\mathcal{A}_m$ associated with this bundle and applied for *wave-forms* being sections of $\bigwedge^k \mathcal{A}_m^* \otimes L_m$. Mathematically, it is a version of the deformation of the de Rham differential considered by Witten [25] and similar to the calculus for *Jacobi algebroids* as developed in [8, 9, 11]. With this calculus, gradients and divergences, so (generalized) Laplace–Beltrami operators, associated with pseudo-Riemannian metrics are naturally defined. This construction, applied to a naturally distinguished pseudo-Riemannian metric on P_m, allows us to write the free Schrödinger operator:

$$\mathcal{S}_m^0\psi = \frac{\hbar^2}{2m}\sum_k \frac{\partial^2\psi}{\partial {y_k}^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial_t},$$

as proportional to the corresponding Laplace–Beltrami operator.

We want to stress three facts. First, we do not just look for transformation rules for solutions of the Schrödinger equation in different reference frames, but we build a bundle, sections of which represent the arguments of the Schrödinger operator ('wavefunctions') that gives to the operator itself a covariant geometrical meaning. Moreover, we show that the projective class of transformation rules, so the projective class of the Schrödinger bundle, is unique. All constructions of this type known to us are based on explicit or hidden assumptions concerning the dynamics of a Newtonian particle, for example assumptions that an intrinsic Lagrangian is a function on the time-configuration-velocity space or that the energy–momentum phase space is the cotangent bundle of the Newtonian spacetime. On the other hand, it has become clear nowadays that an intrinsic, i.e. a frame-independent formulation of the Newtonian dynamics, requires affine and not vectorial objects. We refer here to our earlier work [5, 6, 10, 24] and to recent papers by Janyška and Modugno [13], and Mangiarotti and Sardanashvily [16].

Second, we are able to interpret the standard Schrödinger operator as a (generalized) Laplace–Beltrami operator. To do that one has to use a deformed differential calculus, based on a de Rham-like differential which is similar to the one considered by Witten [25] and similar to the differential in the theory of the so-called *Jacobi algebroids* [8, 9, 11]. In this calculus, the Laplace–Beltrami operator associated with a naturally distinguished invariant pseudo-Riemannian metric on the Schrödinger principal bundle turns out to coincide up to a factor with the classical free (with the potential 0) Schrödinger operator. In our opinion, this idea may find much broader applications than just the ones present in our paper.

Last but not the least, we prove that the proposed formulation is strictly related to the frame-independent formulation of analytical Newtonian mechanics [10]. The 'logarithm' $\mathbf{Z}_m$ of the principal Schrödinger bundle is namely an $\mathbb{R}$-principal bundle, so an *affine values bundle (AV bundle)* in the terminology of [5–7, 10, 24]. The Hamiltonian bundle, i.e. an AV bundle whose sections represent possible Hamiltonians, constructed out of it coincides with the bundle obtained in [6, 10] for the Newtonian particle with mass m. This means that the bundle $\mathbf{Z}_m$ is a Hamilton–Jacobi bundle for the Newtonian particle with mass m, i.e. it is an AV bundle whose sections are subject of the affine (frame-independent) Hamilton–Jacobi equations. This makes a bridge between the classical and quantum theory which, in our opinion, is not yet understood completely and almost not present in the literature. The nice relation of the constructed Schrödinger bundle to the intrinsic Lagrangian or Hamiltonian bundle of a massive Newtonian particle we view as an evidence that our description is proper. In this sense, the present work is a natural step following the series of papers [5–7] in which we have developed the geometry of affine values and applied it to frame-independent formulation of classical mechanics.

The paper is organized as follows. We start with recalling the Newtonian picture for the spacetime and the standard Schrödinger operators associated with potentials on it. Then, we present the main idea of what a 'wavefunction' and the Schrödinger operator should be, and in section 3 we present the idea of a principal or vector bundle with a distinguished set of trivializations.

In section 4, we find the unique form of the transformation rules in the trivial complex line bundle over $\mathbb{R}^3 \times \mathbb{R}$ that leave the Schrödinger operators invariant. These transformation rules are used in constructing the Schrödinger principal $U(1)$ bundle P_m and the Schrödinger line bundle L_m (for fixed 'mass' m). The 'wavefunctions' are understood as sections of L_m and, for every fixed potential U, the Schrödinger operator $\mathbb{S}_m^U$ associated with this potential is a well-defined second-order differential operator on L_m. This description is our frame-independent interpretation of the Schrödinger operators.

In section 5, we show that the above description agrees with the frame-independent description of the Newtonian mechanics and that there is a close relation of the Schrödinger bundles with the affine bundles whose sections are interpreted as a subject of the Hamilton–Jacobi equations and whose phase bundle gives rise to an affine Hamiltonian formalism, as defined in [6, 10].

A differential calculus for wave-forms, i.e. sections of the bundles $\left(\bigwedge^k \mathcal{A}_m^*\right) \otimes_N L_m$, where $\mathcal{A}_m^*$ is the bundle dual to the so-called *Atiyah Lie algebroid* $\mathcal{A}_m$ associated with the principal bundle P_m, is developed in section 6.

Section 7 is devoted to finding a naturally distinguished pseudo-Riemannian metric μ_m on P_m—the *Schrödinger metric*—which, in coordinates associated with any inertial frame, extends the standard spatial Euclidean metrics in the spacetime and which looks exactly in the same way for all inertial observers. We also find the volume form associated with this metric.

The above-mentioned metric and the volume are used in the following section to define the corresponding 'gradient wave-vector fields', associated with 'wavefunctions', and wave-divergences associates with the gradients, so, in turn, the corresponding (generalized) Laplace–Beltrami operator. This operator acts on wavefunctions and coincides, up to a factor, with the free Schrödinger operator we started with.

2. Newtonian spacetime

The *Newtonian spacetime* (some authors prefer to call it *Galilean spacetime*, but we follow the terminology of Benenti [2] and Tulczyjew [20]) is a system (N, τ, g), where N is a four-dimensional affine space for which, say, V is the model vector space, where τ is a non-zero element of V^* and where $g\colon E_0 \to E_0^*$ represents an Euclidean metric on $E_0 = \ker \tau$. The corresponding scalar product reads $\langle v|v'\rangle = (g(v))(v')$ and the corresponding norm $\|v\| = \sqrt{\langle v|v\rangle}$. The elements of the space N represent events. The time elapsed between two events is measured by τ:

$$\Delta t(x, x') = \tau(x - x')$$

and the distance between two simultaneous events is measured by g:

$$d(x, x') = \|x - x'\|.$$

The spacetime N is fibred over the time $\mathbb{T} = N/E_0$, which is a one-dimensional affine space modelled on $\mathbb{R}$.

Let E_1 be an affine subspace of V defined by the equation $\tau(v) = 1$. The model vector space for this subspace is E_0. An element of E_1 represents velocity of a particle. The affine structure of N allows us to associate with an element u of E_1 the family of inertial observers that move in the spacetime with the constant velocity u. In this way we can interpret an element of E_1 also as a *class of inertial reference frames*, while an *inertial reference frame* is understood as a pair $(x_0, u) \in N \times E_1$. For a fixed inertial frame (x_0, u), we can identify N with $E_0 \times \mathbb{R}$ by

$$\Phi_{(x_0,u)} : N \to E_0 \times \mathbb{R}, \qquad x \mapsto ((x - x_0) - \tau(x - x_0)u, \tau(x - x_0)) . \tag{2.1}$$

A change of the inertial reference frame results in the change of this identification and it is represented by

$$\Theta_{(x_0,u)}^{(x_0',u')} = \Phi_{(x_0',u')} \circ \Phi_{(x_0,u)}^{-1} : E_0 \times \mathbb{R} \to E_0 \times \mathbb{R}, \tag{2.2}$$

$$(v,t) \mapsto (v - ((x_0' - x_0) - \tau(x_0' - x_0)u') - (u' - u)t, t - \tau(x_0' - x_0)). \tag{2.3}$$

We can fix orthonormal linear coordinates $y = (y_i) : E_0 \to \mathbb{R}^3$ in E_0 so that $\|v\|^2 = \sum_i y_i^2(v)$. Then, with every inertial frame (x_0, u), we can associate coordinates (y, t) in N, thus V, with $(y,t)(x) = \varphi_{(x_0,u)}(x) = (y(x - x_0 - \tau(x - x_0)u), \tau(x - x_0))$, and the change of coordinates $\theta_{(x_0,u)}^{(x_0',u')}$ corresponding to $\Theta_{(x_0,u)}^{(x_0',u')}$ reads

$$\theta_{(x_0,u)}^{(x_0',u')}(y,t) = \varphi_{(x_0',u')} \circ \varphi_{(x_0,u)}^{-1}(y,t) = (y + w_u + y(v)(t + t_0), t + t_0), \tag{2.4}$$

where $(w_u, t_0) = (y(x_0 - x_0' - \tau(x_0 - x_0')u), \tau(x_0 - x_0')) \in \mathbb{R}^3 \times \mathbb{R}$ are coordinates of $w = x_0 - x_0' \in V$ for the observer (x_0, u) and $y(v) \in \mathbb{R}^3$ are coordinates of $v = u - u' \in E_0$. Note that the maps $\theta_{(x_0,u)}^{(x_0',u')}$ are affine transformations that satisfy the *cocycle condition* $\theta_{(x_0',u')}^{(x_0'',u'')} \circ \theta_{(x_0,u)}^{(x_0',u')} = \theta_{(x_0,u)}^{(x_0'',u'')}$. Thus, we have

$$\begin{aligned}\theta_{(x_0',u')}^{(x_0'',u'')}(y,t) &= \theta_{(x_0,u)}^{(x_0'',u'')} \circ \left(\theta_{(x_0,u)}^{(x_0',u')}\right)^{-1} = (y + w_{u'} + y(v')(t + t_0'), t + t_0') \\ &= (y + w_u' + y(v')(t + t_0') + y(v)t_0', t + t_0'),\end{aligned} \tag{2.5}$$

where (w_u', t_0') are coordinates of $w' = x_0' - x_0''$ for the observer (x_0, u) and $v' = u' - u''$.

3. The Schrödinger operator and principal bundles with trivializations

The classical Schrödinger operator in coordinates $(y, t) \in \mathbb{R}^3 \times \mathbb{R}$, for a particle of mass m and a potential $\widetilde{U} \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$, is a second-order complex differential operator which reads

$$\mathcal{S}_m^{\widetilde{U}}\psi = \frac{\hbar^2}{2m}\sum_k \frac{\partial^2\psi}{\partial y_k^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t} - \widetilde{U}\psi. \tag{3.1}$$

Here, $\sum_k \frac{\partial^2}{\partial y_i^2}$ is clearly the spatial Laplace–Beltrami operator associated with the metric g. The problem is that, if assumed as acting on functions, the Schrödinger operator (3.1) is not invariant with respect to the change of coordinates (2.4) associated with the choice of another inertial frame. On the other hand, by arguments coming from physics, the form of the Schrödinger operator should be independent of the choice of an inertial observer.

The solution we propose is that the Schrödinger operator acts in fact on sections of a certain one-dimensional complex vector bundle L_m over N (we will call it *Schrödinger bundle*) which is trivializable (with a list of distinguished trivializations) but with no canonical trivialization. A change of an observer results not only in a change of coordinates but also in the change of the phase of the wavefunction. Thus, the situation is parallel to the one we encounter in a frame-independent description of the standard Lagrangian in Newtonian mechanics [10].

To be more precise, let us recall that a principal or a vector bundle is defined by an atlas of local identifications of our structure with the trivial ones such that the transition maps respect the structure. One assumes often *a priori* that the atlas is maximal. Here, however, we will understand the given and not maximal atlas as an immanent part of the structure. This is because we want the transition maps to preserve an additional structure. This means that a $U(1)$ *bundle P with trivializations* is understood as a smooth manifold equipped with a family of (global) trivializations $\Psi_\lambda : P \to M \times U(1), \lambda \in \Lambda$, over a manifold M such that the transition maps

$$T_\lambda^{\lambda'} = \Psi_{\lambda'} \circ \Psi_\lambda^{-1} : M \times U(1) \to M \times U(1)$$

are $U(1)$ bundle isomorphisms, i.e. they are of the form

$$T_\lambda^{\lambda'}(x, z) = \left(\theta_\lambda^{\lambda'}(x), \mathrm{e}^{\mathrm{i}\widetilde{F}_\lambda^{\lambda'} \circ \theta_\lambda^{\lambda'}(x)} \cdot z\right), \tag{3.2}$$

where $\widetilde{F}_\lambda^{\lambda'} : M \to \mathbb{R}$ are smooth functions. As a consequence, P carries a unique structure of a principal $U(1)$ bundle over $M_0 = P/U(1)$ and the family $(\Psi_\lambda)_{\lambda\in\Lambda}$ of distinguished trivializations over M defines a family of distinguished sections $(\psi_\lambda)_{\lambda\in\Lambda}$, where $\psi_\lambda : M_0 \to P$ is defined by $\Psi_\lambda(\psi_\lambda(M_0)) = M \times \{1\}$. Note that the *pull-back* of a section ψ of the trivial principal bundle $M \times U(1)$, induced by the transition map $T_\lambda^{\lambda'}$, reads

$$\left(T_\lambda^{\lambda'}\right)^* \psi = \left(\mathrm{e}^{-\mathrm{i}\widetilde{F}_\lambda^{\lambda'}} \cdot \psi\right) \circ \theta_\lambda^{\lambda'}. \tag{3.3}$$

However, as the pull-back reverses the order of composition

$$\left(T_{\lambda'}^{\lambda''} \circ T_\lambda^{\lambda'}\right)^* = \left(T_\lambda^{\lambda'}\right)^* \circ \left(T_{\lambda'}^{\lambda''}\right)^*,$$

we will prefer to use the *push-forwards*, $\left(T_\lambda^{\lambda'}\right)_* = \left(\left(T_\lambda^{\lambda'}\right)^*\right)^{-1}$,

$$\left(T_\lambda^{\lambda'}\right)_* \psi = \left(\mathrm{e}^{\mathrm{i}\widetilde{F}_\lambda^{\lambda'}} \cdot \psi\right) \circ \left(\theta_\lambda^{\lambda'}\right)^{-1} \tag{3.4}$$

instead. It is also clear that multiplying every trivialization Ψ_λ of the $U(1)$ bundle P by a complex number $z_\lambda \in U(1)$ (a phase) will give data for another family of trivializations. More precisely, for $z_0 \in U(1)$ denote by $\widehat{z_0}$ the action of z_0 on the principal $U(1)$ bundle $M \times U(1)$. Then, for any map $\Lambda \ni \lambda \mapsto z_\lambda \in U(1)$, $\widehat{\Psi}_\lambda = \widehat{z_\lambda} \circ \Psi_\lambda$ is another list of trivializations of P. We will say that these principal $U(1)$ bundles with trivializations are *in the same projective class* $[P]$. The same can be repeated for *complex line bundles* with trivializations constructed out of these trivializations of principal $U(1)$ bundles, i.e. for the corresponding associated complex line bundles.

Let us stress the fact that isomorphism of such structures depend on an identification of two distinguished atlases, so that principal bundles with trivializations may not be isomorphic *as bundles with trivializations* even being isomorphic as principal bundles.

Definition 3.1. *By a principal $U(1)$ bundle with trivializations over a manifold M we understand a manifold P together with a map $\Psi : \Lambda \to \mathrm{Diff}(P, M \times U(1))$ from a set Λ to the set of diffeomorphisms $\varphi : P \to M \times U(1), \lambda \mapsto \Psi_\lambda$, such that the transition maps $T_\lambda^{\lambda'} = \Psi_{\lambda'} \circ \Psi_\lambda^{-1} : M \times U(1) \to M \times U(1)$ respect the $U(1)$ bundle structure,*

$$T_\lambda^{\lambda'}(x, z) = \left(\theta_\lambda^{\lambda'}(x), \mathrm{e}^{\mathrm{i}\widetilde{F}_\lambda^{\lambda'} \circ \theta_\lambda^{\lambda'}(x)} \cdot z\right),$$

so that they define a principal $U(1)$ bundle structure on P. A projective morphism of principal $U(1)$ bundles with trivializations (P, Ψ) and $(\widetilde{P}, \widetilde{\Psi})$ consists of a map $j : \Lambda \to \widetilde{\Lambda}$ and a $U(1)$ bundle morphism $J : P \to \widetilde{P}$ such that, for each $\lambda \in \Lambda$,

$$\Xi_\lambda = \widetilde{\Psi}_{j(\lambda)} \circ J \circ (\Psi_\lambda)^{-1} : M \times U(1) \to \widetilde{M} \times U(1)$$

is a morphism of principal $U(1)$ bundles which is of the form

$$\Xi_\lambda(x, z) = (\phi_\lambda(x), z_\lambda \cdot z),$$

i.e. which is constant on $U(1)$ up to a multiplication by the constant $z_\lambda \in U(1)$. We call a projective morphism a morphism if the constants are trivial, $z_\lambda = 1$, i.e. if Ξ_λ is identity on $U(1)$.

A projective morphism as above is a projective isomorphism if the map j is bijective and J is an isomorphism of principal bundles . A projective class $[P]$ of a principal $U(1)$ bundle with trivializations consists of all principal $U(1)$ bundles with trivializations that are projectively isomorphic to (P, Ψ). Again, for isomorphisms of principal $U(1)$ bundles with trivializations, the map j is bijective and J is an isomorphism of principal bundles.

In the above sense, a *trivial bundle* is a bundle with just one trivialization and it is not isomorphic with bundles with the set of trivializations containing more than one element, since there is no way to distinguish one trivialization from another. Moreover, isomorphisms between trivial bundles can be identified with diffeomorphisms between base manifolds. More precisely, they are of the form $\widehat{\phi} : M \times U(1) \to \widetilde{M} \times U(1)$, $\widehat{\phi}(x, z) = (\phi(x), z)$, where $\phi : M \to \widetilde{M}$ is a diffeomorphism.

Theorem 3.1. *$U(1)$-principal bundles with trivializations (P, Ψ) and $(\widetilde{P}, \widetilde{\Psi})$ are isomorphic (respectively, projectively isomorphic) if and only if there is a bijection $j : \Lambda \to \widetilde{\Lambda}$ and a $U(1)$ bundle isomorphism $J : P \to \widetilde{P}$ that relates (relates, up to a constant factor) the distinguished sections ψ_λ and $\widetilde{\psi}_{j(\lambda)}$ for all $\lambda \in \Lambda$.*

Proof. Suppose a bijection $j : \Lambda \to \widetilde{\Lambda}$ and a $U(1)$ bundle isomorphism $J : P \to \widetilde{P}$ define an isomorphism. Since $J \circ (\Psi_\lambda)^{-1} = (\widetilde{\Psi}_{j(\lambda)})^{-1} \circ \Xi_\lambda$,

$$\begin{aligned} J(\psi_\lambda(M_0)) &= J((\Psi_\lambda)^{-1}(M \times \{1\})) = (\widetilde{\Psi}_{j(\lambda)})^{-1}(\Xi_\lambda(M \times \{1\})) \\ &= (\widetilde{\Psi}_{j(\lambda)})^{-1}(\widetilde{M} \times \{1\}) = \widetilde{\psi}_{j(\lambda)}(\widetilde{M}_0), \end{aligned}$$

so the sections ψ_λ and $\widetilde{\psi}_{j(\lambda)}$ are J-related.

Conversely, if ψ_λ and $\widetilde{\psi}_{j(\lambda)}$ are J-related, then

$$J((\Psi_\lambda)^{-1}(M \times \{1\})) = (\widetilde{\Psi}_{j(\lambda)})^{-1}(\widetilde{M} \times \{1\}),$$

so that

$$\Xi_\lambda = \widetilde{\Psi}_{j(\lambda)} \circ J \circ (\Psi_\lambda)^{-1}(x, z) = (\phi_\lambda(x), z)$$

and the trivializations are isomorphic.

The proof in the projective case is analogous. □

The transition maps automatically satisfy the *cocycle condition*

$$T_\lambda^\lambda = id, \qquad T_{\lambda'}^{\lambda''} \circ T_\lambda^{\lambda'} = T_\lambda^{\lambda''}, \tag{3.5}$$

which can be rewritten in the form

$$\theta_\lambda^\lambda = \mathrm{id}, \qquad \theta_{\lambda'}^{\lambda''} \circ \theta_\lambda^{\lambda'} = \theta_\lambda^{\lambda''}, \qquad \widetilde{F}_\lambda^\lambda = 0, \qquad \widetilde{F}_\lambda^{\lambda''} \circ \theta_{\lambda'}^{\lambda''} = \widetilde{F}_{\lambda'}^{\lambda''} \circ \theta_{\lambda'}^{\lambda''} + \widetilde{F}_\lambda^{\lambda'}. \tag{3.6}$$

The cocycle condition can be interpreted as the fact that

$$T : \Lambda \times \Lambda \ni (\lambda', \lambda) \mapsto T_\lambda^{\lambda'} \in \mathrm{Aut}(M \times U(1))$$

is a morphism of the pair groupoid $\Lambda \times \Lambda$ into the group $\mathrm{Aut}(M \times U(1))$ of automorphisms of the principal bundle $M \times U(1)$.

Of course, as easily seen, one can start with transition maps (3.2) satisfying the cocycle condition (3.6) and construct the corresponding principal bundle with trivializations up to isomorphism by taking P to be the space of classes in $\Lambda \times M \times U(1)$ with respect to the equivalence relation

$$[\lambda, x, z] \sim [\lambda', x', z'] \quad \Leftrightarrow \quad T_\lambda^{\lambda'}(x, z) = \left(\theta_\lambda^{\lambda'}(x), \mathrm{e}^{\mathrm{i}\widetilde{F}_\lambda^{\lambda'} \circ \theta_\lambda^{\lambda'}(x)} \cdot z\right) = (x', z'). \tag{3.7}$$

This is canonically a principal $U(1)$ bundle with respect to the action $z_0[\lambda, x, z] = [\lambda, x, z_0 \cdot z]$ with a family Ψ_λ of trivializations indexed by Λ and defined by

$$\Psi_\lambda([\lambda, x, z]) = (x, z) \in M \times U(1).$$

The transition functions for these trivializations coincide with $T_\lambda^{\lambda'}$.

It is completely obvious that the data given by transition maps for a principal $U(1)$ bundle can be used to construct a unique (up to isomorphism) complex line bundle. Our *Schrödinger complex line bundle* L_m (for the mass m) will be obtained as a complex vector bundle with the model fibre $\mathbb{C}$—associated with a principal $U(1)$ bundle P_m with trivializations indexed by inertial observers—the *Schrödinger principal bundle*. Since one often regards wavefunctions as being defined up to a constant phase, it is only the projective class of a $U(1)$ bundle with trivializations that really matters. We will see that all possible Schrödinger bundles are in the same class which means uniqueness of this structure.

Similarly like a principal $U(1)$ bundle with trivializations can be defined up to isomorphism by the family of transition functions $T_\lambda^{\lambda'}$ satisfying the cocycle conditions (3.5), the projective class of a principal $U(1)$ bundle with trivializations can be defined up to projective isomorphism by the family of transition functions $T_\lambda^{\lambda'}$ satisfying the cocycle conditions (3.5) up to constants (we will call such T a *projective cocycle*):

$$T_\lambda^\lambda(x, z) = \widehat{z}_\lambda, \qquad T_{\lambda'}^{\lambda''} \circ T_\lambda^{\lambda'} \circ \left(T_\lambda^{\lambda''}\right)^{-1}(x, z) = \widehat{z}_{(\lambda'', \lambda', \lambda)}. \tag{3.8}$$

Indeed, let us choose λ_0 and define a new family of 'transition functions' $\widetilde{\Psi}_\lambda = T_{\lambda_0}^\lambda$,

$$\widetilde{T}_\lambda^{\lambda'} = T_{\lambda_0}^{\lambda'} \circ \left(T_{\lambda_0}^\lambda\right)^{-1}.$$

Then $\widetilde{T}_\lambda^\lambda = \mathrm{i}d$ and $\widetilde{T}_{\lambda'}^{\lambda''} \circ \widetilde{T}_\lambda^{\lambda'} = \widetilde{T}_\lambda^{\lambda''}$, so the family $\widetilde{T}_\lambda^{\lambda'}$ satisfies the cocycle condition and gives rise to a well-defined principal $U(1)$ bundle with trivializations. If we choose in the above construction another λ_0, say λ_1, then the family of transition maps

$$T_{\lambda_1}^{\lambda'} \circ \left(T_{\lambda_1}^\lambda\right)^{-1}$$

differs from $\widetilde{T}_\lambda^{\lambda'}$ by constant factors, so defines a principal $U(1)$ bundle with trivializations in the same projective class,

Theorem 3.2. *A map $T : \Lambda \times \Lambda \to Aut(M \times U(1))$, $(\lambda', \lambda) \mapsto T_\lambda^{\lambda'}$, satisfying the cocycle condition (3.5) (respectively, the cocycle condition up to constants (3.8)), defines canonically a principal $U(1)$ bundle with trivializations indexed by Λ up to isomorphism (respectively, up to projective isomorphism).*

4. The Schrödinger bundles

The Schrödinger complex line bundle will have trivializations enumerated by inertial observers $\lambda = (x_0, u)$. We have to combine every change of coordinates (2.4) in N with a linear change in values of wavefunctions:

$$T_{(x_0', u')}^{(x_0'', u'')}(y, t, z) = \left(\theta_{(x_0', u')}^{(x_0'', u'')}(y, t), \mathrm{e}^{F_{(x_0', u')}^{(x_0'', u'')} \circ \theta_{(x_0', u')}^{(x_0'', u'')}(y,t)} \cdot z\right), \tag{4.1}$$

so that the push-forward of wavefunctions

$$\left(T_{(x_0', u')}^{(x_0'', u'')}\right)_*(\psi)(y, t) = \mathrm{e}^{F_{(x_0', u')}^{(x_0'', u'')}(y,t)} \cdot \psi\left(\left(\theta_{(x_0', u')}^{(x_0'', u'')}\right)^{-1}(y, t)\right) \tag{4.2}$$

preserves the form of the Schrödinger operator. Of course, as mentioned above, there is an obvious freedom in constructing such a line bundle, as we can always put

$$\widetilde{F}_{(x_0', u')}^{(x_0'', u'')} = F_{(x_0', u')}^{(x_0'', u'')} + A(x_0'', u'') - A(x_0', u')$$

for any function $A : N \times E_1 \to \mathbb{C}$, as the cocycle condition is automatically satisfied and the multiplication by a constant function commutes with the Schrödinger operator. We will see later on that this is the only freedom admitted by our conditions.

At the beginning, we can simplify this problem a little bit. Since, as can be easily seen, the part corresponding to the potential $\widetilde{U}$ associated with a function U on N behaves properly and the Schrödinger operator is invariant with respect to the change of coordinates associated with observers moving with the same velocity, $u = u'$, we can assume that $\widetilde{U} = 0$ and $x_0' = x_0$. Thus, we shall look for an action of the commutative group E_0 in $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}$ of the form

$$R_v(y, t, z) = (y + y(v)t, t, \mathrm{e}^{F_v(y+y(v)t,t)}z), \tag{4.3}$$

corresponding to the representation of E_0 in the algebra $C^\infty_{\mathbb{C}}(\mathbb{R}^3 \times \mathbb{R})$ of complex-valued functions on $\mathbb{R}^3 \times \mathbb{R}$,

$$(R_v)_*(\psi)(y, t) = \mathrm{e}^{F_v(y,t)}\psi(y - y(v)t, t), \tag{4.4}$$

such that the 'free' Schrödinger operator

$$\mathcal{S}^0_m\psi = \frac{\hbar^2}{2m}\sum_k \frac{\partial^2\psi}{\partial y_i^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t} \tag{4.5}$$

remains unchanged:

$$\mathcal{S}^0_m(\mathrm{e}^{F_v(y,t)}\psi(y - y(v)t, t)) = \mathrm{e}^{F_v(y,t)}\mathcal{S}^0_m(\psi)(y - y(v)t, t). \tag{4.6}$$

Remark 4.1. That our spatial part is three dimensional is motivated by physics. However, from the mathematical point of view, there is no difference if we use other dimensions. All considerations and proofs remain unchanged if we use $\mathbb{R}^n \times \mathbb{R} \times \mathbb{C}$ instead of $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}$.

Let us look at what the function F_v should be, in order that (4.6) is satisfied. Straightforward calculations, where we put for simplicity $y(v) = v = (v_k)$, show that (4.6) is equivalent to

$$\begin{aligned}\psi(y - vt, t)&\left(\mathrm{i}(\partial_t F_v)(y, t) + \frac{\hbar}{2m}\left(\sum_k \left(\partial_{y_k} F_v\right)^2(y, t) + \sum_k \left(\partial^2_{y_k} F_v\right)(y, t)\right)\right)\\ &+ \sum_k \left(\partial_{y_k}\psi\right)(y - vt, t)\left(\frac{\hbar}{m}\left(\partial_{y_k} F_v\right)(y, t) - \mathrm{i}v_k\right) = 0\end{aligned} \tag{4.7}$$

for all complex functions ψ on $\mathbb{R}^3 \times \mathbb{R}$. Since ψ is arbitrary, this, in turn, is equivalent to the system of equations

$$\mathrm{i}(\partial_t F_v)(y, t) + \frac{\hbar}{2m}\left(\sum_k \left(\partial_{y_k} F_v\right)^2(y, t) + \sum_k \left(\partial^2_{y_k} F_v\right)(y, t)\right) = 0, \tag{4.8}$$

$$\frac{\hbar}{m}\left(\partial_{y_k} F_v\right)(y, t) - \mathrm{i}v_k = 0, \qquad k = 1, 2, 3. \tag{4.9}$$

From (4.9) it follows that $\partial^2_{y_k} F_v = 0, k = 1, 2, 3$, so that (4.8) reduces to

$$\mathrm{i}(\partial_t F_v)(y, t) - \frac{m}{2\hbar}\sum_k v_k^2 = 0. \tag{4.10}$$

Equations (4.9) and (4.10) for partial derivatives determine F_v up to a constant, so, as can be easily seen,

$$F_v(y, t) = \frac{\mathrm{i}m}{\hbar}\left(\sum_k v_k y_k - \frac{t}{2}\sum_k v_k^2\right) + c. \tag{4.11}$$

Going back to the general case, we conclude that the transformation rule (4.2) that preserves the form of the Schrödinger operator requires that

$$F^{(x_0'',u'')}_{(x_0',u')}(y,t) = \frac{\mathrm{i}m}{\hbar}\left(\sum_k v_k' y_k - \frac{t}{2}\sum_k (v_k')^2\right) + c^{(x_0'',u'')}_{(x_0',u')}.$$

The cocycle condition

$$T^{(x_0'',u'')}_{(x_0',u')} \circ T^{(x_0',u')}_{(x_0,u)} = T^{(x_0'',u'')}_{(x_0,u)} \tag{4.12}$$

now yields that

$$F^{(x_0'',u'')}_{(x_0',u')} = F^{(x_0'',u'')}_{(x_0,u)} - F^{(x_0',u')}_{(x_0,u)} \circ \left(\theta^{(x_0'',u'')}_{(x_0',u')}\right)^{-1},$$

i.e.

$$c^{(x_0'',u'')}_{(x_0',u')} + c^{(x_0',u')}_{(x_0,u)} = c^{(x_0'',u'')}_{(x_0,u)} + \sum_k \left((w'_{u'})_k - \frac{t_0'}{2}v_k\right)v_k. \tag{4.13}$$

If we take another family of constants

$$\widetilde{c}^{(x_0'',u'')}_{(x_0',u')} = c^{(x_0'',u'')}_{(x_0',u')} + d^{(x_0'',u'')}_{(x_0',u')},$$

then (4.13) implies

$$d^{(x_0'',u'')}_{(x_0',u')} + d^{(x_0',u')}_{(x_0,u)} = d^{(x_0'',u'')}_{(x_0,u)}. \tag{4.14}$$

But, as easily seen, the only functions d on an affine finite-dimensional space that satisfy (4.14) are of the form

$$d^{(x_0',u')}_{(x_0,u)} = A(x_0',u') - A(x_0,u)$$

for certain function A, i.e. we get only the obvious freedom in constructing the line bundle. Thus, we get the following.

Theorem 4.1. *Let us fix a class of inertial observers $u \in E_1$. The transformations (4.2) respect the Schrödinger operator (4.5) and satisfy the cocycle condition (4.12) if and only if the functions $F^{(x_0'',u'')}_{(x_0',u')}$ are of the form*

$$F^{(x_0'',u'')}_{(x_0',u')}(y,t) = \frac{\mathrm{i}m}{\hbar}\left(\sum_k \left(y_k - \frac{t}{2}v_k'\right)v_k' + \sum_k\left((w'_{u'})_k - \frac{t_0'}{2}v_k\right)v_k\right) + A(x_0'',u'') - A(x_0',u'), \tag{4.15}$$

for $w'_{u'} = y((x_0'-x_0'') - \tau(x_0'-x_0'')u')$ being the coordinates of $x_0'-x_0'' \in V$ with respect to the inertial observer (x_0',u'), for $t_0' = \tau(u'-u'')$, for $v' = (v_k')$ being the coordinates of $u'-u'' \in E_0$, for $v = (v_k)$ being the coordinates of $u-u' \in E_0$, and A being an arbitrary function $A : N \times E_1 \to \mathbb{C}$.

Remark 4.2. The fact that certain transformations of the form (4.4) act on solutions of the Schrödinger equation in different reference frames is known (see e.g. [17, p 100] or [4, section 4.3]). Here, we have found a general form of such transformations in order to properly recognize the arguments of the Schrödinger operator. Moreover, such transformations have been proven to be unique up to the obvious freedom.

Removing constants from (4.15), we will stay in the same of projective class of the corresponding principal $U(1)$ bundle. Thus, we get the following.

Theorem 4.2. *There is a unique projective class $\mathbf{P}_m$ of principal $U(1)$ bundles P_m over the Newtonian spacetime with trivializations $\Psi_{(x_0,u)} : P_m \to \mathbb{R}^3 \times \mathbb{R} \times U(1)$ indexed by inertial observers $(x_0, u) \in N \times E_1$ and covering the coordinate maps on the base*

$$(y, t)(x) = \varphi_{(x_0,u)}(x) = (y(x - x_0 - \tau(x - x_0)u), \tau(x - x_0))$$

such that the transition maps

$$T^{(x_0',u')}_{(x_0,u)} = \Psi_{(x_0',u')} \circ \left(\Psi_{(x_0,u)}\right)^{-1} : \mathbb{R}^3 \times \mathbb{R} \times U(1) \to \mathbb{R}^3 \times \mathbb{R} \times U(1)$$

leave the Schrödinger operator $\mathcal{S}^0_m$ invariant. This projective class is represented by the projective cocycle

$$\mathcal{T}^{(x_0',u')}_{(x_0,u)}(y, t, z) = \left(y + v(t + t_0) + w_u, t + t_0, \mathrm{e}^{\frac{\mathrm{i}m}{\hbar}(\langle y,v\rangle + \frac{t}{2}\|v\|^2)} \cdot z\right), \tag{4.16}$$

where $v \in \mathbb{R}^3$ are coordinates of $u - u' \in E_0$ and $(w_u, t_0) = (y(x_0 - x_0' - \tau(x_0 - x_0')u), \tau(x_0 - x_0'))$ are coordinates of $x_0 - x_0'$ for any inertial observer (x_0, u) in the class of u.

We call any representative of the class $\mathbf{P}_m$ a *Schrödinger principal bundle* and the corresponding complex line bundle L_m the *Schrödinger line bundle*.

According to theorem 4.1, the differential operator $\mathbb{S}^{(x_0',u')}_m$ on L_m, which corresponds to $\mathcal{S}^0_m$ on the trivial one-dimensional vector bundle $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{C}$ *via* the trivialization $\Psi_{(x_0',u')}$, does not depend on the trivialization, so it gives rise to a well-defined differential operator $\mathbb{S}^0_m$ on L_m. Choosing a potential $U \in C^\infty_{\mathbb{C}}(N)$, we can write the full Schrödinger operator as $\mathbb{S}^U_m\psi = \mathbb{S}^0_m\psi + U\psi$ acting on sections of L_m. We can summarize these observations as follows.

Theorem 4.3. *For any function (potential) U on the Newtonian spacetime N there is a well-defined (trivialization-independent) differential operator $\mathbb{S}^U_m$ (the Schrödinger operator), acting on sections of the Schrödinger line bundle L_m. This operator corresponds, via the trivialization $\Psi_{(x_0,u)}$, to the differential operator*

$$\mathcal{S}^m_U\psi = \frac{\hbar^2}{2m}\sum_k \frac{\partial^2\psi}{\partial y_i^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t} - \left(U \circ \varphi^{-1}_{(x_0,u)}\right)\psi \tag{4.17}$$

acting on complex functions $\psi(y, t)$ on $\mathbb{R}^3 \times \mathbb{R}$.

A Schrödinger principal bundle P_m can be, for example, constructed according to the general scheme (3.7). Let us fix $u \in E_1$ and put in (4.15) $A = 0$. Then the transition maps corresponding to the phase change F can be written in the form

$$\begin{aligned} T^{(x_0'',u'')}_{(x_0',u')}(y, t, z) &= \left(\theta^{(x_0'',u'')}_{(x_0',u')}(y, t), \mathrm{e}^{F^{(x_0'',u'')}_{(x_0',u')}(\theta^{(x_0'',u'')}_{(x_0',u')}(y,t))} \cdot z\right) \\ &= \left(y + w'_{u'} + (t + t_0')v', t + t_0', \exp\left(\frac{\mathrm{i}m}{\hbar}\left(\left\langle y + w'_{u'} + \frac{1}{2}(t + t_0')v', v'\right\rangle\right.\right.\right. \\ &\quad \left.\left.\left. + \left\langle w'_{u'} + \frac{t_0'}{2}v, v\right\rangle\right)\right) \cdot z\right), \end{aligned} \tag{4.18}$$

where $w'_{u'} = y((x_0' - x_0'') - \tau(x_0' - x_0'')u')$, $t_0' = \tau(u' - u'')$, $v' = y(u' - u'')$ and $v = y(u - u')$. The set P^u_m of equivalence classes of the relation

$$(x_0', u', y', t', z') \sim (x_0'', u'', y'', t'', z'') \iff T^{(x_0'',u'')}_{(x_0',u')}(y', t', z') = (y'', t'', z'') \tag{4.19}$$

defined on the product $N \times E_1 \times \mathbb{R}^3 \times \mathbb{R} \times U(1)$ is a principal $U(1)$ bundle over N with the projection

$$[x_0', u', y', t', z'] \longmapsto x_0' + y^{-1}(y') + t'u' = \varphi^{-1}_{(x_0',u')}(y', t') \in N.$$

For each inertial observer (x_0', u') in each equivalence class of the relation $\sim$, there is one representative with (x_0', u') in the first two places. It means that we have a mapping

$$\Psi_{(x_0',u')} : P_m^u \ni [x_0', u', y', t', z'] \longmapsto (y', t', z') \in (\mathbb{R}^3 \times \mathbb{R} \times U(1))$$

which is the trivialization (over $\mathbb{R}^3 \times \mathbb{R}$) of P_m^u corresponding to the inertial observer (x_0', u') and

$$\Psi_{(x_0'',u'')} \circ \Psi^{-1}_{(x_0',u')} = T^{(x_0'',u'')}_{(x_0',u')},$$

so the pair $\left(P_m^u, \Psi\right)$ is a principal bundle with trivialization which is a representative of the class $\mathbf{P}_m$.

Remark 4.3. Of course, as solving a concrete Schrödinger equation always takes place in a given coordinate system, introducing the concept of the Schrödinger bundle does not imply new methods in finding the solutions. It just gives a geometrical structure capturing the necessary gauging of the wavefunctions while passing from one inertial frame to another. All the geometrical setting supports the idea that wavefunctions should be understood as classes $[\psi]$ not feeling a change by a constant phase. On the principal Schrödinger bundle such a class is represented by an invariant horizontal foliation, so by a flat principal connection. It is interesting that in this setting, one can associate with a class of inertial observers moving with velocity v with respect to a given one a plane wave

$$W_v(y,t) = \exp\left[\frac{\mathrm{i}m}{\hbar}\left(\sum_k v_k y_k - \frac{t}{2}\sum_k v_k^2\right)\right].$$

We should multiply a wavefunction by this plane wave, so change its phase by the phase of this plane wave, before writing the wavefunctions in coordinates associated with the new observer. In this sense, for quantum systems, different inertial observers carry not only relative velocities but also relative plane waves.

5. Relation to Newtonian mechanics

By means of a group homomorphism

$$\mathbb{R} \to U(1)\colon s \mapsto \exp\left(\frac{\mathrm{i}s}{\hbar}\right), \tag{5.1}$$

the Schrödinger principal $U(1)$ bundle P_m can be considered as the reduced principal $(\mathbb{R}, +)$ bundle $\mathbf{Z}_m$ and the 'additive projective class' of $\mathbf{Z}_m$ does not depend on the choice of P_m. For direct calculation we can use the bundle $\mathbf{Z}_m^u$, the 'logarithm' of P_m^u, with trivializations transforming according to

$$\begin{aligned}\bar{T}^{(x_0'',u'')}_{(x_0',u')}(y,t,s) = \bigg(& y + w'_{u'} + (t+t_0')v', t+t_0', s + m\left\langle y + w'_{u'} + \frac{1}{2}(t+t_0')v', v'\right\rangle \\ &+ m\left\langle w'_{u'} + \frac{t_0'}{2}v, v\right\rangle\bigg).\end{aligned} \tag{5.2}$$

It is an AV bundle in terminology of [6]. Analogously as in (4.19), an element of $\mathbf{Z}_m^u$ is an equivalence class of $(x_0', u', y', t', s') \in N \times E_1 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$

$$(x_0', u', y', t', s') \sim (x_0'', u'', y'', t'', s'') \iff \bar{T}^{(x_0'',u'')}_{(x_0',u')}(y', t', s') = (y'', t'', s''), \tag{5.3}$$

and the projection $\zeta : \mathbf{Z}_m^u \to N$ on N reads

$$[x_0', u', y', t', s'] \longmapsto x_0' + y^{-1}(y') + t'u' = \varphi^{-1}_{(x_0',u')}(y', t') \in N.$$

Since N is fibred over affine time, $\bar{\tau} : N \to \mathbb{T}$, the standard construction of the Hamiltonian AV bundle [6, 10, 24] yields

$$\mathsf{Ph}_\zeta : \mathsf{Ph}\big(\mathbf{Z}^u_m\big) \to \underline{\mathsf{Ph}\big(\mathbf{Z}^u_m\big)},$$

where $\mathsf{Ph}\big(\mathbf{Z}^u_m\big)$ is the phase bundle of the AV bundle $\mathbf{Z}^u_m$ and

$$\underline{\mathsf{Ph}\big(\mathbf{Z}^u_m\big)} = \mathsf{Ph}\big(\mathbf{Z}^u_m\big)\big/\langle \mathrm{d}t\rangle$$

(see [6, 7, 10, 24]). Using a trivialization, we can identify the above fibration with

$$\mathsf{Ph}_\zeta : \mathsf{T}^*(\mathbb{R}^3 \times \mathbb{R}) \to (\mathsf{T}^*\mathbb{R}^3 \times \mathbb{R})/\langle \mathrm{d}t\rangle.$$

The transition maps (5.2) act on sections σ, represented in the trivializations by functions $\sigma = \sigma(y, t)$, as

$$\big(\bar{T}^{(x''_0,u'')}_{(x'_0,u')}\big)_*(\sigma)(y,t) = \sigma(y - tv' - w'_{u'}, t - t'_0) + m\left(\left\langle y - \frac{t}{2}v', v'\right\rangle + \left\langle w'_{u'} - \frac{t'_0}{2}v, v\right\rangle\right),$$

so the adapted Darboux coordinates in $\mathsf{T}^*(\mathbb{R}^3 \times \mathbb{R})$ transform according to

$$\mathsf{Ph}\big(\bar{T}^{(x''_0,u'')}_{(x'_0,u')}\big)(y, t, p_y, p_t) = \left(y + w'_{u'} + (t + t'_0)v', t + t'_0, p_y + mv', p_t - \langle p_y, v'\rangle - \frac{m}{2}\|v'\|^2\right). \tag{5.4}$$

Since, by convention, the distinguished vertical vector field on the Hamiltonian AV bundle is $-\partial_{p_t}$, the vertical coordinate—value of Hamiltonian sections—is $h = -p_t$ and in coodinates (y, t, p_y, h) we get $\mathsf{Ph}_\zeta(y, t, p_y, h) = (y, t, p_y)$, and the transition maps in the form

$$\mathsf{Ph}\big(\bar{T}^{(x''_0,u'')}_{(x'_0,u')}\big)(y, t, p_y, h) = \left(y + w'_{u'} + (t + t'_0)v', t + t'_0, p_y + mv', h + \langle p_y, v'\rangle + \frac{m}{2}\|v'\|^2\right).$$

Note that these transformations do not depend on the distinguished $u \in E_1$ nor x'_0, x''_0 any longer but only on the relative velocity $v' = u' - u''$, so the Hamiltonian bundle $\mathbf{H}_m = \mathsf{Ph}\big(\mathbf{Z}^u_m\big)$ does not depend on u and in fact on the choice of P_m in the projective class $\mathbf{P}_m$. In this bundle, during transitions, the momenta (as elements of E^*_0) transform according to the rule $p \mapsto p + m\langle v', \cdot\rangle$, and the values of possible Hamiltonian sections, according to the rule $h \mapsto h + \langle p, v'\rangle + \frac{m}{2}\|v'\|^2$, which is precisely the transformation used in [6, 10] to define the Hamiltonian AV bundle for a Newtonian particle of mass m. This means that the AV bundle $\mathbf{Z}^u_m$ plays the role of the affine Hamilton–Jacobi bundle: the Hamilton–Jacobi equation is an equation of sections σ of $\mathbf{Z}^u_m$. This bundle, however, is not uniquely determined. If $\mathbf{d}\sigma : N \to \mathsf{Ph}\big(\mathbf{Z}^u_m\big) = \mathbf{H}_m$ denotes the affine de Rham differential, then the Hamilton–Jacobi equation associated with the Hamiltonian section $h : \underline{\mathbf{H}_m} \to \mathbf{H}_m$ takes the form

$$\mathbf{d}\sigma(N) \subset h(\underline{\mathbf{H}_m}).$$

In coordinates, this Hamilton–Jacobi equation takes the standard form

$$h\left(y, t, \frac{\partial\sigma}{\partial y}\right) + \frac{\partial\sigma}{\partial t}(y,t) = 0.$$

6. Atiyah bundle and generalized differential calculi

Let us fix a principal Schrödinger bundle P_m. If we use the parametrization

$$\mathbb{R} \ni r \mapsto \exp\left(-\frac{\mathrm{i}m}{\hbar}r\right) \in U(1) \tag{6.1}$$

of $U(1)$, then the change of coordinates (4.18) in P_m associated with the change of inertial frames reads

$$T^{(x_0'',u'')}_{(x_0',u')}(y,t,r) = \left(\theta^{(x_0'',u'')}_{(x_0',u')}(y,t), r - \frac{\hbar}{\mathrm{i}m} F^{(x_0'',u'')}_{(x_0',u')}\left(\theta^{(x_0'',u'')}_{(x_0',u')}(y,t)\right)\right)$$
$$= \left(y + w'_{u'} + (t+t_0')v', t+t_0', r - \left\langle y + w'_{u'} + \frac{1}{2}(t+t_0')v', v'\right\rangle - \left\langle w'_{u'} + \frac{t_0'}{2}v, v\right\rangle\right). \tag{6.2}$$

Let us now observe that every smooth section $\psi : N \to P_m$ gives rise to a smooth complex function $\widetilde{\psi}$ on P_m defined by

$$\widetilde{\psi}\left(\exp\left(\frac{\mathrm{i}m}{\hbar}r\right)\cdot\psi(x)\right) = \exp\left(\frac{\mathrm{i}m}{\hbar}r\right), \tag{6.3}$$

for any $n \in N$. In coordinates associated with a choice of an inertial frame,

$$\widetilde{\psi}(y,t,r) = \mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}\psi(y,t). \tag{6.4}$$

We can use the same local formula to produce the function $\widetilde{\psi}$ on P_m also from a section ψ of the Schrödinger complex line bundle L_m associated with P_m:

$$\widetilde{\psi}\left(\exp\left(\frac{\mathrm{i}m}{\hbar}r\right)\cdot\frac{\psi(x)}{|\psi(x)|}\right) = \exp\left(\frac{\mathrm{i}m}{\hbar}r\right)|\psi(x)|,$$

if $\psi(x) \neq 0$ and $\widetilde{\psi} = 0$ on the fibre over x otherwise. Note that the 'absolute value' $|\psi(x)|$ is well defined on L_m, since it is a complex line bundle associated with an $U(1)$ principal bundle. Moreover, the principal bundle P_m can be considered to be the set of unitary elements of L_m.

The functions of the form $\widetilde{\psi}$ on P_m are characterized as $\frac{\mathrm{i}m}{\hbar}$ homogeneous functions with respect to the fundamental vector field ∂_r of the $U(1)$ action. Indeed, if $\partial_r(f) = \frac{\mathrm{i}m}{\hbar}f$, then the function ψ written in our coordinates as $\mathrm{e}^{-\frac{\mathrm{i}mr}{\hbar}}f$ represents a section of the Schrödinger bundle L_m. To see this, note first of all that $\psi = \psi(y,t)$ does not depend on r. Second, under the change of coordinates (6.2)

$$f(y,t,r) = \psi(y,t)\,\mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}$$

is pushed forward into

$$f\circ\left(T^{(x_0'',u'')}_{(x_0',u')}\right)^{-1}(y,t,r) = f\left(\left(\theta^{(x_0'',u'')}_{(x_0',u')}\right)^{-1}(y,t), r + \frac{\hbar}{\mathrm{i}m}F^{(x_0'',u'')}_{(x_0',u')}(y,t)\right)$$
$$= \psi\circ\left(\theta^{(x_0'',u'')}_{(x_0',u')}\right)^{-1}(y,t)\cdot\mathrm{e}^{F^{(x_0'',u'')}_{(x_0',u')}(y,t)}\cdot\mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}. \tag{6.5}$$

But (6.5) is $\psi'(y,t)\,\mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}$, where ψ' is the push-forward of ψ, so ψ is pushed forward according to the rule

$$\psi \mapsto \mathrm{e}^{F^{(x_0'',u'')}_{(x_0',u')}}\cdot\psi\circ\left(\theta^{(x_0'',u'')}_{(x_0',u')}\right)^{-1},$$

i.e. exactly like sections of the Schrödinger bundle do. Thus, we get the following.

Theorem 6.1. *The local formula* $\widetilde{\psi}(y,t,r) = \psi(y,t)\,\mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}$ *establishes a one-to-one correspondence between sections* ψ *of the Schrödinger line bundle* L_m *and* $\frac{\mathrm{i}m}{\hbar}$ *homogeneous (with respect to the fundamental vector field* ∂_r*) functions on* P_m.

Remark 6.1. The above correspondence between sections of the Schrödinger principal $U(1)$ bundle P_m and functions on P_m is similar to the analogous correspondence between sections of an AV bundle $\mathbf{A}$ and functions on $\mathbf{A}$ as exploited in [5–7]. The latter can be viewed as a 'classical' counterpart of this correspondence for $U(1)$-principal bundles with trivializations (see the following section). The function $\widetilde{\psi}$ on P_m obtained from a section ψ of the bundle L_m coincides with a function on P_m obtained from ψ by viewing at the associated line bundle L_m as the reduced trivial bundle $\mathbb{C} \times P_m$.

Let us consider now the complex *Atiyah bundle* $\mathcal{A}_m$ over N associated with the principal $U(1)$ bundle P_m. Let us also recall that the Atiyah bundle can be characterized as the vector bundle over the base of the principal G-bundle P whose sections are represented by G-invariant vector fields on P. In our case, we choose vector fields with complex coefficients which make no real difference. As such vector fields are projectable, we have a canonical surjective bundle map $\rho : \mathcal{A}_m \to \mathsf{T}N$ (the *anchor map*) with the kernel KP_m. Moreover, since invariant vector fields are closed with respect to the Lie bracket, we have a canonical *Lie algebroid* structure on $\mathcal{A}_m$—the *Atiyah Lie algebroid* of P_m. For a detailed description of Lie and Atiyah algebroids, we refer to the monograph [15, section 3.2]. The sections of $\mathcal{A}_m$ in our case are represented by complex vector fields on P_m, commuting with the fundamental vector field ∂_r of the $U(1)$-action, and the Lie algebroid bracket is represented by the commutator of vector fields. In coordinates associated with a trivialization of P_m, they are of the form

$$X = \sum_k f_k(y,t)\partial_{y_k} + g(y,t)\partial_t + h(y,t)\partial_r. \tag{6.6}$$

Every such invariant vector field—section of $\mathcal{A}_m$—can be canonically interpreted, in turn, as a first-order differential operator D_X on the Schrödinger complex line bundle. Indeed, as such a vector field commutes with ∂_r, it acts on $\frac{im}{\hbar}$ homogeneous functions $\widetilde{\psi}$, so sections ψ of L_m by

$$\widetilde{(D_X(\psi))} = X(\widetilde{\psi}).$$

Since

$$X(\widetilde{\psi}) = X\big(\psi \cdot \mathrm{e}^{\frac{imr}{\hbar}}\big) = \left(\sum_k f_k(y,t)\frac{\partial\psi}{\partial y_k}(y,t) + g(y,t)\frac{\partial\psi}{\partial t}(y,t) + \frac{\mathrm{i}m}{\hbar}h(y,t)\psi(y,t)\right)\mathrm{e}^{\frac{imr}{\hbar}},$$

the section (6.6) of $\mathcal{A}_m$ represents in coordinates the first-order differential operator

$$D_X = \sum_k f_k(y,t)\partial_{y_k} + g(y,t)\partial_t + \frac{\mathrm{i}m}{\hbar}h(y,t), \tag{6.7}$$

acting on sections of the Schrödinger complex line bundle L_m. It is easy to see that the Lie algebroid structure on $\mathcal{A}_m$ is represented by the standard commutator of differential operators. Note however that, as there is no canonical trivialization of L_m, the space of sections does not carry a canonical structure of an associative algebra, so derivations are not distinguished. We will call the sections of $\mathcal{A}_m$ *Schrödinger vector fields*. In general, we will call tensor fields built out of $\mathcal{A}_m$ *Schrödinger tensor fields*. They are represented by invariant tensor fields on the Schrödinger principal bundle P_m. In particular, *Schrödinger k-forms* are sections of $\bigwedge^k \mathcal{A}_m^*$ and they are represented by $U(1)$-invariant k-forms on P_m. However, if for a given trivialization $\Psi_{(x_0,u)}$ we interpret the functional coefficients of a tensor field as *wavefunctions*—sections of L_m—we get *wave tensor fields*, i.e. sections of the corresponding tensor bundle of $\mathcal{A}_m$ tensored (over N) with L_m. In particular, *wavefunctions* are sections of L_m, *wave-forms* are sections of $\left(\bigwedge^k \mathcal{A}_m^*\right) \otimes_N L_m$ and *wave-vector fields* are sections of $\mathcal{A}_m \otimes L_m$. Under transition maps, the wave-tensor fields transform with a change in phases exactly like wavefunctions.

We can extend the observation of theorem 6.1 to wave-tensor fields.

Theorem 6.2. *The formula* $\widetilde{\omega}(y,t,r) = \omega(y,t)\,\mathrm{e}^{\frac{imr}{\hbar}}$, *expressed in coordinates associated with a distinguished trivialization* $\Psi_{(x_0,u)}$, *establishes a one-to-one correspondence between* *wave-tensor fields ω and $\frac{\mathrm{i}m}{\hbar}$ homogeneous (with respect to the fundamental vector field ∂_r) tensor fields $\widetilde{\omega}$ on the Schrödinger principal bundle P_m. This local formula depends on the trivialization.*

On the wave-forms we have an analog $\widetilde{\mathrm{d}}$ of the standard de Rham differential d, defined by

$$\widetilde{(\widetilde{\mathrm{d}}\omega)} = \mathrm{d}\widetilde{\omega}.$$

Of course, by definition, $\widetilde{\mathrm{d}}^2 = 0$. In coordinates associated with a choice of an inertial frame, this differential reads

$$\widetilde{\mathrm{d}}\omega = \mathrm{d}\omega + \frac{\mathrm{i}m}{\hbar}\mathrm{d}r \wedge \omega. \tag{6.8}$$

We will call it *wave-de Rham differential*. We hope these explanations make clear that the contraction of a wave-vector field with a k-covariant Schrödinger tensor is a $(k-1)$-covariant wave-tensor, as the contraction of a $\frac{\mathrm{i}mr}{\hbar}$ homogeneous vector field with an $U(1)$-invariant k-covariant tensor is a $\frac{\mathrm{i}mr}{\hbar}$ homogeneous covariant $(k-1)$-tensor.

Remark 6.2. The local form of the wave-de Rham differential is a particular case of a deformation of the de Rham differential considered already by Witten [25], $\widetilde{\mathrm{d}}\omega = \mathrm{e}^{-\frac{\mathrm{i}mr}{\hbar}} \cdot \mathrm{d}\left(\mathrm{e}^{\frac{\mathrm{i}mr}{\hbar}}\omega\right)$ and generalized to *Jacobi algebroids* (generalized Lie algebroids) in [8, 9, 11].

7. Schrödinger metrics

Consider now a pseudo-Riemannian metric $\mu_m^{(x_0,u)} \in \mathsf{Sec}(\mathcal{A}_m^* \otimes \mathcal{A}_m^*)$ on the Schrödinger principal bundle P_m such that $\mu_m^{(x_0,u)}$ corresponds *via* the trivialization $\Psi_{(x_0,u)} : P_m \to \mathbb{R}^3 \times \mathbb{R} \times U(1)$ (associated with an inertial frame $(x_0,u) \in N \times E_1$) to a pseudo-Riemannian $U(1)$-invariant metric μ on $\mathbb{R}^3 \times \mathbb{R} \times U(1)$ which extends the standard spatial Euclidean metric on $\mathbb{R}^3 \times \mathbb{R}$, i.e. to a metric μ of the form

$$\mu(y,t,r) = \sum_k \mathrm{d}y_k \otimes \mathrm{d}y_k + \sum_k B_k(y,t)\,\mathrm{d}y_k \vee \mathrm{d}r + C(y,t)\,\mathrm{d}r \otimes \mathrm{d}r + D(y,t)\,\mathrm{d}t \vee \mathrm{d}r. \tag{7.1}$$

If we assume additionally that μ is invariant with respect to the change of coordinates (6.2), then $\mu_m = \mu_m^{(x_0,u)}$ is a pseudo-Riemannian metric on P_m which does not depend on the choice of trivialization. We will call such metric μ_m the *Schrödinger metric*. Since μ is $U(1)$-invariant, looking for Schrödinger metrics, we can forget about shifts in the coordinate r and look for μ which is invariant with respect to all maps

$$(y,t,r) \mapsto \left(y + (t+t_0)v + w, t + t_0, r - \sum_k v_k\left(y_k + \frac{t}{2}v_k\right)\right).$$

Straightforward calculations show that B_k and C must be 0, and $D = 1$. Thus, we get the following.

Theorem 7.1. *There is a unique Schrödinger metric μ_m on P_m. In coordinates associated with any bundle trivializations $\Psi_{(x_0,u)}$, it is given by*

$$\mu_m = \sum_k \mathrm{d}y_k \otimes \mathrm{d}y_k + (\mathrm{d}t \otimes \mathrm{d}r + \mathrm{d}r \otimes \mathrm{d}t)\,. \tag{7.2}$$

It is easy to see that the contravariant form of the Schrödinger metric μ_m in coordinates reads

$$\nu_n = \sum_k \partial_{y_k} \otimes \partial_{y_k} + (\partial_t \otimes \partial_r + \partial_r \otimes \partial_t) \,. \tag{7.3}$$

A ν-orthogonal basis of 1-forms is for example $\mathrm{d}y_k, \beta_+, \beta_-$, where $\mathrm{d}y_k$ and $\beta_+ = \frac{\mathrm{d}r+\mathrm{d}t}{\sqrt{2}}$ have length 1 and $\beta_- = \frac{\mathrm{d}r-\mathrm{d}t}{\sqrt{2}}$ has squared length -1. Therefore, the *Schrödinger volume* Ω_m associated with the Schrödinger metric μ_m (and defined up to a sign) is represented by

$$\Omega_m = \mathrm{d}y \wedge \beta_+ \wedge \beta_- = \mathrm{d}y \wedge \mathrm{d}t \wedge \mathrm{d}r, \tag{7.4}$$

where $\mathrm{d}y = \mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \mathrm{d}y_3$.

Remark 7.1. The metric μ can be transported to a metric on the total space of a Hamilton–Jacobi bundle $\mathbf{Z}^u_m$. The total space of $\mathbf{Z}^u_m$ is an affine space and, for $m = 1$, the metric satisfies the properties of a Galilei metrics postulated in [21]. Thus, $\mathbf{Z}^u_1$ is an example of a Galilei space. A wavefunction on Galilei space (without potential) satisfies the Laplace equation for the Galilei metric and is $\frac{\mathrm{i}m}{\hbar}$ homogeneous. This shows full compatibility of our four-dimensional approach with the wave mechanics of the Galilei space.

8. Schrödinger–Laplace operators for the Schrödinger metrics

With the use of the Schrödinger differential $\widetilde{\mathrm{d}}$ and the Schrödinger metric μ_m, one can define the *wave-gradient* ∇_ψ of a wavefunction ψ—a section of the Schrödinger complex line bundle L_m—in the standard way

$$\mathrm{i}_{\nabla_\psi}\mu_m = \widetilde{\mathrm{d}}\psi. \tag{8.1}$$

The wave-gradient is clearly a wave-vector field. In coordinates,

$$\widetilde{\mathrm{d}}\psi = \sum_k \frac{\partial\psi}{\partial y_k}\mathrm{d}y_k + \frac{\partial\psi}{\partial t}\mathrm{d}t + \frac{\mathrm{i}m}{\hbar}\psi\,\mathrm{d}r$$

and

$$\nabla_\psi = \sum_k \frac{\partial\psi}{\partial y_k}\partial_{y_k} + \frac{\mathrm{i}m}{\hbar}\psi\,\partial_t + + \frac{\partial\psi}{\partial t}\partial_r,$$

where the functional coefficients should be understood as wavefunctions.

For every wave-vector field Y, in turn, its *wave-divergence* $\mathrm{div}(Y)$—associated with the Schrödinger metric μ_m—is defined via the Schrödinger volume Ω_m, like classically, as

$$\mathrm{div}(Y)\Omega_m = \widetilde{\mathrm{d}}(\mathrm{i}_Y\Omega_m). \tag{8.2}$$

Here $i_Y\Omega_m$; thus $\widetilde{\mathrm{d}}(i_Y\Omega_m)$ is a wave-form, as well as the obviously defined product of the wavefunction $\mathrm{div}(Y)$ and the Schrödinger volume form Ω_m. In coordinates,

$$\mathrm{div}\left(\sum_k f_k\partial_{y_k} + g\partial_t + h\partial_r\right) = \sum_k \frac{\partial f_k}{\partial y_k} + \frac{\partial g}{\partial t} + \frac{\mathrm{i}m}{\hbar}h.$$

Finally, we can define the *Schrödinger–Laplace operator* Δ_m, associated with the Schrödinger metric μ_m, by the formula completely analogous to the formula defining standard Laplace–Beltrami operators:

$$\Delta_m\psi = \mathrm{div}(\nabla_\psi). \tag{8.3}$$

The Schrödinger–Laplace operator is therefore a second-order differential operator acting on the Schrödinger complex line bundle L_m, i.e. mapping wavefunctions into wavefunctions. The above definition is completely intrinsic and natural. In coordinates associated with the choice of an inertial frame,

$$\Delta_m \psi = \sum_k \frac{\partial^2 \psi}{\partial {y_k}^2} + \frac{2\mathrm{i}m}{\hbar}\frac{\partial \psi}{\partial t}. \tag{8.4}$$

But this is exactly the free Schrödinger operator $\mathbb{S}^0_m$ on L_M up to a constant factor:

$$\mathbb{S}^0_m \psi = \frac{\hbar^2}{2m}\Delta_m \psi = \frac{\hbar^2}{2m}\sum_k \frac{\partial^2 \psi}{\partial {y_k}^2} + \mathrm{i}\hbar\frac{\partial \psi}{\partial t}.$$

Example 8.1. Consider for simplicity (1 + 1)-dimensional spacetime and inertial frames differing only by the relative velocity $v \in \mathbb{R}$. For fixed mass $m > 0$, with the relative velocity v we associate the plane wave $W_v(y,t)$ on $\mathbb{R}\times\mathbb{R}$ with coordinates (y,t) by

$$W_v(y,t) = \exp\left[\frac{\mathrm{i}m}{\hbar}\left(yv - \frac{t}{2}v^2\right)\right].$$

The Schrödinger line bundle L_m in this setting can be interpreted as quotient $\widetilde{L}/\sim_\pi$ of the trivial complex line bundle $\widetilde{L} = E_1 \times \mathbb{R}\times\mathbb{R}\times\mathbb{C}$, where E_1 is the affine $\mathbb{R}$ (no 0 chosen), modulo the action of the additive group $\mathbb{R}$ acting on $\widetilde{L}$ by $\mathbb{R} \ni v \mapsto \pi_v$,

$$\pi_v(u,y,t,z) = (u+v, y+vt, t, [W_v(y+vt,t)]^{-1}\cdot z). \tag{8.5}$$

This line bundle is associated with the Schrödinger $U(1)$-principal bundle P_m obtained as the quotient of the trivial $U(1)$-principal bundle $\widetilde{P} = E_1 \times \mathbb{R}\times\mathbb{R}\times U(1)$ modulo the $\mathbb{R}$-action completely analogous to (8.5). The sections ψ of L_m (respectively, P_m) are therefore interpreted as sections of $\widetilde{L}$ (respectively, $\widetilde{P}$), $z = \psi(u,y,t)$, which are invariant with respect to this $\mathbb{R}$-action. Hence, for fixed $u \in E_1$, they are viewed as complex valued (respectively, $U(1)$-valued) functions on $\mathbb{R}\times\mathbb{R}$. With a section of L_m represented by $z = \psi(u,y,t)$ we associate the function $\widetilde{\psi}$ on P_m represented by function $\widetilde{\psi}(u,y,t,z) = z\cdot\psi(u,y,t)$ on $\widetilde{P}$ which is simultaneously $\mathbb{R}$-invariant and $U(1)$-invariant. Conversely, every bi-invariant complex-valued function on $\widetilde{P}$ represents a section of L_m in the above way. The differential operator

$$\widetilde{D}_m = \partial_y^2 + \frac{2\mathrm{i}m}{\hbar}\partial_t$$

is clearly $U(1)$-invariant. It is also, $\mathbb{R}$-invariant, $\widetilde{D}_m(f\circ\pi_v) = \widetilde{D}_m(f)\circ\pi_v$ (which is less trivial but straightforward), so it induces a frame-independent differential operator D_m on sections of L_m. When fixing $u \in E_1$, we get the standard free Schrödinger operator

$$\mathbb{S}^0_m \psi = \frac{\hbar^2}{2m}D_m\psi = \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} + \mathrm{i}\hbar\frac{\partial\psi}{\partial t}.$$

But the differential operator $\widetilde{D}_m$ acts on $U(1)$-invariant functions $\widetilde{\psi}(u,y,t,z) = z\cdot\psi(u,y,t)$ on $\widetilde{P}$ as the operator

$$\widetilde{\Delta}_m = \partial_y^2 + \frac{2\mathrm{i}mz}{\hbar}\partial_t\partial_z,$$

which is the Laplace–Beltrami operator of the pseudo-Riemannian metric μ_m represented by the bi-invariant symmetric form

$$\mu = \mathrm{d}y\otimes\mathrm{d}y + \frac{\mathrm{i}\hbar\bar{z}}{m}(\mathrm{d}t\otimes\mathrm{d}z + \mathrm{d}z\otimes\mathrm{d}t).$$

The operator $\widetilde{\Delta}_m$ is the extended Schrödinger operator in the sense of Lizzi–Marmo–Sparano–Vinogradov [14].

9. Concluding remarks

We have found a proper geometrical setting for frame-independent understanding of the classical Schrödinger operators on the Newtonian spacetime and we have found a description of the free Schrödinger operator as a (generalized) Laplace–Beltrami operator.

In this picture, the Schrödinger operators act not on functions on the spacetime but on sections of a certain one-dimensional complex vector bundle—*Schrödinger line bundle*. This line bundle has trivializations indexed by inertial observers and is closely related to an $U(1)$-principal bundle with an analogous list of trivializations—*Schrödinger principal bundle*. If an inertial frame is fixed, the Schrödinger bundle can be identified with the trivial bundle over spacetime, but as there is no canonical trivialization (inertial frame) these sections, interpreted as wavefunctions, cannot be viewed as actual functions on the spacetime. A change of an observer results not only in a change of coordinates, but also in the change of the phase of the wavefunction.

The projective class of all possible Schrödinger bundles is uniquely determined and its 'logarithm' is an $\mathbb{R}$-principal bundle whose sections are a subject of Hamilton–Jacobi equations that makes a bridge between the classical and quantum theory.

On the *Schrödinger principal bundle*, a natural (generalized) differential calculus is developed based on a de Rham-like differential, similar to the one considered by Witten [25] and to the differential of the so-called *Jacobi algebroids* [8, 9, 11]. In this calculus, the (generalized) Laplace–Beltrami operator associated with a naturally distinguished invariant pseudo-Riemannian metric on the Schrödinger principal bundle turns out to coincide, up to a factor, with the classical free Schrödinger operator.

The presented framework is conceptually four dimensional (the base is identified with the traditional Newtonian spacetime, but the values of wavefunctions are not true numbers), does not involve any ad hoc or axiomatically introduced geometrical structures and is based only on the traditional understanding of the Schrödinger operator in a given reference frame. This makes it mathematically simple, demonstrative and respecting the postulate of Occam's Razor.

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